Journey to the World of Numbers through Complex Geometry

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Abstract The speaker has long been interested in applications of complex geometry to number theory, and will trace the trajectory of his involvement concerning (1) abelian schemes over quasi-projective varieties, (2) commutants of Hecke correspondences on bounded symmetric domains Ω, and (3) functional transcendence concerning Shimura varieties and more generally $X_\Gamma := \Omega/\Gamma$ of finite volume.

For (1) we recall (a) first results of Mok (1991) and Mok-To (1993) concerning the finiteness of Mordell-Weil groups of universal abelian varieties $\mathbf{A}_\Gamma$ without fixed parts over modular function fields $K = \mathcal{M}(\overline{X_\Gamma})$, (b) upper bounds of Mok (1991) on $\mathrm{rank}(\mathbf{A}(K))$, $K = \mathcal{M}(\overline{X})$, for abelian varieties $\mathbf{A}$ obtained from $\mathbf{A}_\Gamma$ through finite dominating classifying maps $f : X \to X_\Gamma$, and (c) in connection to the study of the Betti map on elliptic schemes: the finiteness of points of Betti multiplicities ≥ 2 for a section $\sigma \in \mathbf{E}(\mathcal{M}(\overline{X}))$ of infinite order due to Corvaja-Demeio-Masser-Zannier (2022) and its effective version due to Ulmer-Urzúa (2021), and a differential-geometric proof of the latter by Mok-Ng (2022). For (2) we recall (d) a problem of Clozel-Ullmo (2003) concerning commutants of Hecke correspondences which reduces to a conjecture characterizing measure-preserving germs of holomorphic maps on Ω, and the solution of Mok-Ng (2012) using CR geometry in conjunction with Alexander's theorem for the complex unit ball $\mathbb{B}^n$, $n \geq 2$, together with its generalization by Mok-Ng to the higher-rank situation. For (3) we explain (e) the compactification theorem of Mok-Zhong (1989) in Kähler geometry applicable to X_Γ, (f) the Ax-Lindemann theorem of Mok (2019) for arbitrary lattices in the case of $\mathbb{B}^n$, (g) the Ax-Schanuel theorem of Mok-Pila-Tsimerman (2019) for Shimura varieties, and (h) the characterization of bi-algebraicity due to Chan-Mok (2022) in the case of a projective variety $Y \subset X_{\check{\Gamma}}$, for $X_{\check{\Gamma}}$ possibly of infinite volume, uniformized by an algebraic subset $Z \subset \Omega$.